OCR
Oxford Cambridge and RSA

Monday 18 October 2021 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours

8980408492

INSTRUCTIONS
- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION
- This Insert contains the article for Section B.
- This document has **4** pages.

© OCR 2021 [603/1002/9]
DC (LK/SG) 300393/4
OCR is an exempt Charity

Adding arctangents

Where does the name 'arctangent' come from?

The two commonly used ways to denote the angle which has a tangent x are $\tan^{-1}x$ and $\arctan x$. The first of these is related to inverse function notation, $f^{-1}(x)$. Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle; $\arctan x$ is the arc whose tangent is x.

An interesting result

It can be shown that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

Consider the diagram in **Fig. C1**.

Triangle ABC is right-angled at B.
AB = BC = 1 cm.
D is the midpoint of BC.

Using triangle ABD, $\tan\alpha=\frac{DB}{BA}=\frac{1}{2}$ so $\alpha=\arctan\left(\frac{1}{2}\right)$.

Using triangle ABC, $\tan(\alpha+\beta)=1$ so $\alpha+\beta=\arctan 1$.

Hence $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1$.

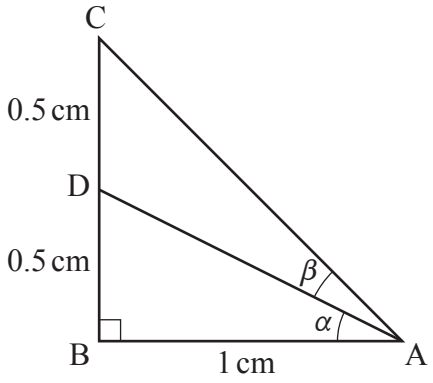


Fig. C1

Using $\tan\alpha=\frac{1}{2}$ and finding $\tan\beta$, it follows that $\beta=\arctan\left(\frac{1}{3}\right)$,

which gives the required result that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

Generalising the result

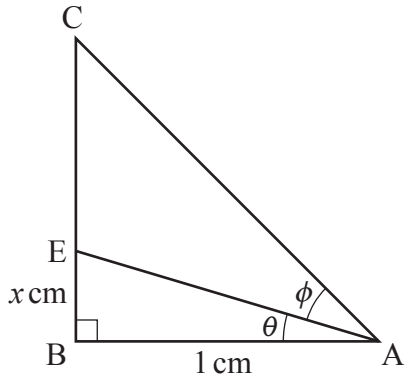


Fig. C2

Triangle ABC in **Fig. C2** is the same as triangle ABC in **Fig. C1** but E is a point on BC such that EB = x cm and $\theta=\arctan x$.

Following the same method as above, $\arctan x+\arctan\left(\frac{1-x}{1+x}\right)=\arctan 1$.

The arctan addition formula

The arctangent addition formula is a further generalization:

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$, as long as $xy < 1$.

This result is equivalent to the addition formula $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ where $\alpha = \arctan x$ and $\beta = \arctan y$.

To see why the restriction $xy < 1$ is necessary, consider what happens if $xy \geqslant 1$.

Clearly, $\frac{x+y}{1-xy}$ is undefined when $xy = 1$, so the formula does not apply in this case.

Suppose next that $xy > 1$, and that x and y are both positive; in this case $y > \frac{1}{x}$.

For any positive x, $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

$y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$ so it follows that $\arctan x + \arctan y > \frac{\pi}{2}$.

However, $\arctan\left(\frac{x+y}{1-xy}\right)$ cannot be greater than $\frac{\pi}{2}$ as the range of the arctan function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The formula $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ therefore cannot be valid in this case.

A similar argument can be used to show that the formula cannot be valid when $xy > 1$ and x and y are both negative.

If $xy > 1$, the arctangent addition formula needs to be adapted, as shown below.

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi$, when $xy > 1$ and $x, y < 0$

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$, when $xy > 1$ and $x, y > 0$

Some additional results

- For n a positive integer, $\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right)$; this follows directly from the arctan addition formula in line 23.
- $\arctan 1 + \arctan 2 + \arctan 3 = \pi$. This can be proved by using $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ together with $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1$.

© OCR 2021

OCR
Oxford Cambridge and RSA

Copyright Information

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website (www.ocr.org.uk) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of the Cambridge Assessment Group; Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

© OCR 2021